THE KERNELS AND CONTINUITY IDEALS OF HOMOMORPHISMS FROM $\mathcal{C}_0(\Omega)$

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In memoriam: Graham Robert Allan, 1937-2007

Abstract. We give a description of the continuity ideals and the kernels of homomorphisms from the algebras of continuous functions on locally compact spaces into Banach algebras.

1. Introduction

Let $\theta : A \to B$ be a homomorphism from a commutative Banach algebra A into a Banach algebra B. The *continuity ideal* of θ is defined to be the ideal

$$\mathcal{I}(\theta) = \{a \in A : \text{ the map } b \mapsto \theta(ab),\ A \to B,\ \text{ is continuous}\};$$

this ideal contains every ideal I in A on which θ is continuous. Moreover, in the case where $A = \mathcal{C}_0(\Omega)$, for a locally compact space Ω, then θ is continuous on $\mathcal{I}(\theta)$.

This is a continuation of [17]. Here, we aim to characterize the ideals which are the kernels or the continuity ideals of homomorphisms from $\mathcal{C}_0(\Omega)$ into Banach algebras. This is, in some sense, a last piece of the picture of homomorphisms from $\mathcal{C}_0(\Omega)$ into Banach algebras. So what do we know about these objects so far?

Denoted by $|\cdot|_\Omega$ the uniform norm on Ω. *For brevity, we shall call a homomorphism into a radical Banach algebra a radical homomorphism.*

It is a theorem of Kaplansky [15] that, for each algebra norm $\|\cdot\|$ on $\mathcal{C}_0(\Omega)$ and each $f \in \mathcal{C}_0(\Omega)$, we have $\|f\| \geq |f|_\Omega$. This essentially gives the description of all continuous homomorphisms from $\mathcal{C}_0(\Omega)$ into Banach algebras. It was immediately asked [15] whether discontinuous homomorphisms from $\mathcal{C}_0(\Omega)$ exist. In 1970s, this question was resolved in the positive independently by Dales [4] and Esterle [8], [9], [10]. Moreover, they showed that, assuming the Continuum Hypothesis (CH), for each (non-compact) locally compact space Ω and each non-modular prime ideal P in $\mathcal{C}_0(\Omega)$ with $|\mathcal{C}_0(\Omega)/P| = \mathfrak{c}$, there exists a radical homomorphism from $\mathcal{C}_0(\Omega)$ with kernel precisely equal to P. (For more details see [5].)

Preceding this resolution was Bade and Curtis's theorem [2] which shows that each discontinuous homomorphism from $\mathcal{C}_0(\Omega)$ into a Banach algebra B can be decomposed into a sum of a continuous homomorphism and a finite number of discontinuous linear maps, each of which is a homomorphism into the radical of B when restricted to a maximal ideal of $\mathcal{C}_0(\Omega)$. The following statement of the

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theorem also includes some improvements from [8] and [18] (see also [5], [19]); see §2 for notations.

Theorem 1.1. *Let Ω be a locally compact space, and let θ be a discontinuous homomorphism from $\mathcal{C}_0(\Omega)$ into a Banach algebra B. Suppose that $\theta(\mathcal{C}_0(\Omega))$ is dense in B.*

(i) *The continuity ideal $\mathcal{I}(\theta)$ is the largest ideal of $\mathcal{C}_0(\Omega)$ on which θ is continuous.*

(ii) *There exists a non-empty finite subset $\{p_1, \ldots, p_n\}$ of $\Omega^\flat$ such that*

$$\bigcap_{i=1}^{n} J_{p_i} \subset \mathcal{I}(\theta) \subset \bigcap_{i=1}^{n} M_{p_i}.$$

(iii) *There exists a continuous homomorphism $\mu : \mathcal{C}_0(\Omega) \to B$ such that*

$$B = \mu(\mathcal{C}_0(\Omega)) \oplus \operatorname{rad} B, \quad \mu(\bigcap_{i=1}^{n} M_{p_i}) \cdot \operatorname{rad} B = \{0\}\,,$$

and $\mu = \theta$ on a dense subalgebra of $\mathcal{C}_0(\Omega)$ containing $\mathcal{I}(\theta)$.

(iv) *Set $\nu = \theta - \mu$. Then ν maps into $\operatorname{rad} B$, and the restriction of ν to $\bigcap_{i=1}^{n} M_{p_i}$ is a homomorphism ν' onto a dense subalgebra of $\operatorname{rad} B$ such that $\mathcal{I}(\theta) = \ker \nu'$.*

(v) *There exist linear maps $\nu_1, \ldots, \nu_n : \mathcal{C}_0(\Omega) \to \operatorname{rad} B$ such that*
   (a) *$\nu = \nu_1 + \cdots + \nu_n$,*
   (b) *each $\nu_i | M_{p_i}$ $(1 \le i \le n)$ is a non-zero radical homomorphism, and*
   (c) *$\nu_i(\mathcal{C}_0(\Omega)) \cdot \nu_j(\mathcal{C}_0(\Omega)) = \{0\}$ for each $1 \le i \ne j \le n$.*

(vi) *The ideals $\ker \theta$ and $\mathcal{I}(\theta)$ are always intersections of prime ideals.* □

In particular, this result emphasizes the roles of prime ideals and of radical homomorphisms as building blocks for general discontinuous homomorphisms from $\mathcal{C}_0(\Omega)$: If we know about the radical homomorphisms from $\mathcal{C}_0(\Omega)$ we will know about the homomorphisms from $\mathcal{C}_0(\Omega)$ into Banach algebras. Dales and Esterle's theorem shows how to construct radical homomorphisms from $\mathcal{C}_0(\Omega)$ with kernel being finite intersection of (non-modular) prime ideals. In fact, for some spaces Ω, the kernels of radical homomorphisms from $\mathcal{C}_0(\Omega)$ are always finite intersections of prime ideals ([8], [17]).

However, in [17], we show that for most metrizable non-compact locally compact spaces Ω, for example $\mathbb{R}$, there exists a radical homomorphism from $\mathcal{C}_0(\Omega)$ whose kernel is not the intersection of any finite number of prime ideals.

In this paper, we shall show that the kernel of a radical homomorphism from $\mathcal{C}_0(\Omega)$ is always the intersection of a relatively compact family of (non-modular) prime ideals. We also prove that, assuming the Continuum Hypothesis (CH), under a minor cardinality condition, when restricted to those ideals that are intersections of countably many prime ideals, the kernels of radical homomorphisms from $\mathcal{C}_0(\Omega)$ are exactly the intersections of relatively compact family of non-modular prime ideals in $\mathcal{C}_0(\Omega)$. Similar result holds for continuity ideals of homomorphisms from $\mathcal{C}_0(\Omega)$ into Banach algebras. (See §6.)

Remark. The Continuum Hypothesis is required in construction of discontinuous homomorphisms from $\mathcal{C}_0(\Omega)$ into Banach algebras, for it has been proved by Solovay and Woodin that it is relatively consistent with ZFC that all such homomorphisms are continuous (see [6] for the proof and references).

2. Preliminary definitions and notations

Let A be a commutative algebra. The (*conditional*) *unitization* $A^{\#}$ of A is defined as the algebra A itself if A is unital, and as A with identity adjoined otherwise. The identity of $A^{\#}$ is denoted by $\mathbf{e}_A$.

A prime ideal or semiprime ideal in A must be a proper ideal. However, we consider A itself as a finite intersection of prime ideals (*the intersection of the empty collection of prime ideals*).

Define the *prime radical* $\sqrt{I}$ of an ideal I in A to be the intersection of all prime ideals in A containing I, so that

$$\sqrt{I} = \{a \in A : \ a^n \in I \text{ for some } n \in \mathbb{N}\} \,.$$

For each ideal I in A and each element $a \in A^{\#}$, define the *quotient of I by a* to be the ideal

$$I{:}a = \{b \in A : \ ab \in I\} \,.$$

Clearly we have $I \subset I{:}a$ in each case.

Let I be an ideal in $A^{\#}$. A subset S of $A^{\#}$ is *algebraically independent modulo I* if $p(a_1, \dots, a_n) \notin I$ for each $n \in \mathbb{N}$, each non-zero polynomial $p \in \mathbb{C}[X_1, \dots, X_n]$, and each n-tuple $(a_1, \dots, a_n)$ of distinct elements of S. A *transcendence basis* for $A^{\#}$ modulo I is a maximal set among all the subsets of $A^{\#}$ which are algebraically independent modulo I; such a basis always exists.

For an well-ordered set Λ, we denote by $\mathrm{o}(\Lambda)$ the ordinal that is order isomorphic to Λ.

For the definition of *universal* algebras, see [5, Definition 5.7.8]. The important fact that we need is the existence of universal radical Banach algebras. For example, the integral domain $L^1(\mathbb{R}^+, \omega)$ is universal for each radical weight ω bounded near the origin [5, Theorem 5.7.25]. Indeed, the class of universal commutative radical Banach algebras has been characterized in [12] (see also [5, Theorem 5.7.28]).

Let B be a Banach algebra, and let S be an indexing set. Define $\ell^\infty(S, B)$ to be the Banach algebra of all bounded families $(b_\alpha : \alpha \in S)$ in B under pointwise algebraic operations and the supremum norm.

For a discussion of the theory of the algebras of continuous functions, see [5], [7] or [13]. Here, we just give some facts which are needed in our discussion.

Let Ω be a locally compact space; the convention is that locally compact spaces and compact spaces are Hausdorff. The *one-point compactification* of Ω is denoted by $\Omega^\flat$. Denote by $\mathcal{C}_c(\Omega)$ the *algebra of compactly supported continuous functions* on Ω. For each $p \in \Omega$, define

$$\begin{aligned} J_p &= \{f \in \mathcal{C}_0(\Omega) : \ f \text{ is zero on a neighbourhood of } p\}\,, \\ M_p &= \{f \in \mathcal{C}_0(\Omega) : \ f(p) = 0\}\,. \end{aligned}$$

For p being the point (at infinity) adjoined to Ω to obtain $\Omega^\flat$, we also set

$$J_p = \mathcal{C}_c(\Omega) \quad \text{and} \quad M_p = \mathcal{C}_0(\Omega).$$

For each prime ideal P in $\mathcal{C}_0(\Omega)$, there always exists a unique point $p \in \Omega^\flat$ such that $J_p \subset P \subset M_p$, we say that P *is supported at the point* p. It can be seen that P is modular if and only if its support point belongs to Ω.

It is an important fact that, for each prime ideal P in $\mathcal{C}_0(\Omega)$, the set of prime ideals in $\mathcal{C}_0(\Omega)$ which contain P is a chain with respect to the inclusion relation.

For each function f continuous on Ω, the *zero set* of f is denoted by $\mathbf{Z}(f)$. The set of zero sets of continuous functions on Ω is denoted by $\mathbf{Z}[\Omega]$.

A *z-filter* $\mathcal{F}$ on Ω is a non-empty proper subset of $\mathbf{Z}[\Omega]$ satisfying:

(i) $Z_1 \cap Z_2$ belongs to $\mathcal{F}$ whenever both Z_1 and Z_2 belong to $\mathcal{F}$,
(ii) if $Z_1 \in \mathcal{F}$, $Z_2 \in \mathbf{Z}[\Omega]$ and $Z_1 \subset Z_2$, then Z_2 also belongs to $\mathcal{F}$.

Each z-filter $\mathcal{F}$ corresponds to an ideal

$$\{f \in \mathcal{C}(\Omega) : \ \mathbf{Z}(f) \in \mathcal{F}\},$$

denoted by $\mathbf{Z}^{-1}[\mathcal{F}]$; each such ideal is called a *z-ideal*.

3. Relatively compact families of prime ideals

In this section, let A be a commutative algebra.

Definition 3.1 (cf. [17] Definition 3.1)**.** An indexed family $(P_i)_{i\in S}$ of prime ideals in A is *pseudo-finite* if $a \in P_i$ for all but finitely many $i \in S$ whenever $a \in \bigcup_{i\in S} P_i$.

For a pseudo-finite sequence (P_n) of prime ideals, it is obvious that $\bigcup_{n=1}^{\infty} P_n$ is either a prime ideal in A or the whole A.

Definition 3.2. A family $\mathfrak{C}$ of prime ideals in A is *relatively compact* if every sequence of prime ideals in $\mathfrak{C}$ contains a pseudo-finite subsequence. The family $\mathfrak{C}$ is *compact* if it is relatively compact and contains the union of each of its pseudo-finite sequence.

Obviously, the union of finitely many pseudo-finite families is relatively compact. In the rest of this section, we shall justify our choice of terminology.

Denote by Π the set of prime ideals in A. For $a_1, a_2, \dots, a_m$, and b in A, define

$$\mathcal{U}^b_{a_1,\dots,a_m} = \{P \in \Pi : \ a_i \in P \ (1 \le i \le m) \text{ and } b \notin P\}.$$

Then the collection of all $\mathcal{U}^b_{a_1,\dots,a_m}$'s is a base for a topology τ. Indeed, by the primeness, we have

$$\mathcal{U}^{bd}_{a_1,\dots,a_m,c_1,\dots,c_n} = \mathcal{U}^b_{a_1,\dots,a_m} \cap \mathcal{U}^d_{c_1,\dots,c_n}.$$

Moreover, we *claim* that τ is Hausdorff. For, let $P_1 \neq P_2 \in \Pi$. If $P_1 \not\subset P_2$ and $P_2 \not\subset P_1$, say $a \in P_1 \setminus P_2$ and $b \in P_2 \setminus P_1$, then $P_1 \in \mathcal{U}^b_a$, $P_2 \in \mathcal{U}^a_b$, and $\mathcal{U}^b_a \cap \mathcal{U}^a_b = \emptyset$. Otherwise, say $P_1 \subsetneq P_2$, choose $b \in P_2 \setminus P_1$ and $c \in A \setminus P_2$, then $P_1 \in \mathcal{U}^b_0$, $P_2 \in \mathcal{U}^c_b$, and $\mathcal{U}^b_0 \cap \mathcal{U}^c_b = \emptyset$.

Next, we *claim* that $\mathcal{U}^u_0$ is τ-compact ($u \in A$). Indeed, we see that $\{\mathcal{U}^u_a, \ \mathcal{U}^a_0 : \ a \in A\}$ is a subbasis for the relative τ-topology on $\mathcal{U}^u_0$, so by Alexander's lemma, we need only to show that each cover of $\mathcal{U}^u_0$ by sets in this subbasis has a finite subcover. So, let E, F be subsets of A such that

$$\mathcal{U}^u_0 = \bigcup_{a\in E} \mathcal{U}^u_a \cup \bigcup_{b\in F} \mathcal{U}^b_0.$$

Set $S = \{u^m a_1 \cdots a_n : \ m, n \in \mathbb{N}, \ a_1, \dots, a_n \in E\}$, and let I be the ideal generated by F. Assume toward a contradiction that $S \cap I = \emptyset$. Then since S is closed under multiplication, there exists a prime ideal P such that $P \supset I$ and $P \cap S = \emptyset$; this implies that $P \in \mathcal{U}^u_0$ but $P \notin \bigcup_{a\in E} \mathcal{U}^u_a \cup \bigcup_{b\in F} \mathcal{U}^b_0$, a contradiction. Thus, $S \cap I \neq \emptyset$,

so there exist $k, m, n \in \mathbb{N}$, $a_1, \ldots, a_m \in E$ and $b_1, \ldots, b_n \in F$, and $c_1, \ldots, c_n \in A$ such that $u^k a_1 \cdots a_m = b_1 c_1 + \cdots b_n c_n$. We can then deduce that

$$\mathcal{U}_0^u = \bigcup_{i=1}^m \mathcal{U}_{a_i}^u \cup \bigcup_{j=1}^n \mathcal{U}_0^{b_j}.$$

Thus τ is locally compact.

The one point compactification of (Π, τ) can be considered as the set $\Pi \cup \{A\}$; a basis of neighbourhood for A is given by

$$\mathcal{U}_{b_1,\ldots,b_n} = \{A\} \cup \{P \in \Pi : \ b_i \in P \ (1 \le i \le n)\}.$$

Proposition 3.3. *Let A be a commutative algebra. Denote by Π the set of prime ideals in A. Define a topology τ as above. Then (Π, τ) is a totally disconnected locally compact space, and every [relative] compact family of prime ideals in A is a [relatively] sequentially τ-compact subset of $\Pi \cup \{A\}$.*

Proof. It remains to prove the last assertion. We *claim* that a pseudo-finite sequence (P_n) of prime ideals in A is τ-convergent in $\Pi \cup \{A\}$. In fact, set $P = \bigcup_{n=1}^{\infty} P_n$. Then either $P \in \Pi$ or $P = A$. In both cases, we can check that (P_n) τ-converges to P. $\square$

Remark. Suppose that A is unital. Then $\Pi = \mathcal{U}_0^{\mathbf{e}_A}$, and so (Π, τ) is compact. In the above proposition, we can replace $(\Pi \cup \{A\}, \tau)$ by (Π, τ).

Remark. Suppose that $A = \mathcal{C}_0(\Omega)$ for some locally compact space Ω. Then for each pseudo-finite sequence (P_n) of prime ideals in $\mathcal{C}_0(\Omega)$, the union $\bigcup_{n=1}^{\infty} P_n$ is in fact a prime ideal in $\mathcal{C}_0(\Omega)$. Again, in the above proposition, we can replace $(\Pi \cup \{A\}, \tau)$ by (Π, τ).

Remark. Let us instead consider a topology σ on $\Pi \cup \{A\}$ generated by

$$\mathcal{U}_{a_1,\ldots,a_m}^Q = \{P \in \Pi \cup \{A\} : \ P \subset Q \text{ and } a_i \in P \ (1 \le i \le m)\},$$

where Q is either a semiprime ideal in A or A itself, and $a_1, \ldots, a_m \in Q$. Then a sequence of prime ideals in A is pseudo-finite if and only if it is convergent in $(\Pi \cup \{A\}, \sigma)$, and so a family of prime ideals in A is [relatively] compact if and only if it is [relatively] sequentially compact in $(\Pi \cup \{A\}, \sigma)$. However, in general, σ is neither Hausdorff nor locally compact. In the case where $A = \mathcal{C}_0(\Omega)$, then $(\Pi \cup \{A\}, \sigma)$ is Hausdorff (but not locally compact).

4. Abstract continuity ideals and relatively compact families of prime ideals

Let $\theta : A \to B$ be a homomorphism from a commutative Banach algebra A into a Banach algebra B. Let $(a_n : \ n \in \mathbb{N})$ be a sequence in A. Then

$$\mathcal{I}(\theta){:}a_1 a_2 \cdots a_n \subset \mathcal{I}(\theta){:}a_1 a_2 \cdots a_{n+1} \qquad (n \in \mathbb{N}).$$

It follows easily from the stability lemma (see [5, 5.2.7] or [19, 1.6] for the statement and proof) that there exists n_0 such that

$$\mathcal{I}(\theta){:}a_1 a_2 \cdots a_n = \mathcal{I}(\theta){:}a_1 a_2 \cdots a_{n+1} \qquad (n \ge n_0).$$

Thus $\mathcal{I}(\theta)$ is an *abstract continuity ideal* in the following sense.

Definition 4.1. Let A be a commutative algebra. An ideal I is an *abstract continuity ideal* if, for each sequence (a_n) in A, there exists n_0 such that

$$I{:}a_1a_2\cdots a_n = I{:}a_1a_2\cdots a_{n+1} \qquad (n \geq n_0).$$

Proposition 4.2. *Let $\mathfrak{P}$ be a relatively compact family of prime ideals in a commutative algebra A. Then $\bigcap\{P : P \in \mathfrak{P}\}$ is an abstract continuity ideal in A.*

Proof. Denote by I the intersection $\bigcap\{P : P \in \mathfrak{P}\}$. Assume toward a contradiction that I is not an abstract continuity ideal. Then there exists a sequence (a_n) in A such that

$$I{:}a_1a_2\cdots a_n \subsetneq I{:}a_1a_2\cdots a_{n+1} \qquad (n \in \mathbb{N}).$$

For each n, we see that

$$I{:}a_1\cdots a_n = \bigcap\{P \in \mathfrak{P} :\ a_1\cdots a_n \notin P\}.$$

Thus, it follows that, there exists $P_n \in \mathfrak{P}$ such that $a_1\cdots a_n \notin P_n$ but $a_1\cdots a_{n+1} \in P_n$. The relative compactness implies that there exists $n_1 < n_2 < \cdots$ such that (P_{n_i}) is pseudo-finite. However, we see that $a_1\cdots a_{n_2} \in P_{n_1}$, but $a_1\cdots a_{n_2} \notin P_{n_i}$ $(i \geq 2)$; this contradicts the pseudo-finiteness. □

The remaining of this section is devoted to a converse of the above proposition. *Let I be an abstract continuity ideal of a commutative algebra A. Denote by $\mathfrak{P}$ the set of prime ideals of the form $I{:}a$ for some $a \in A$.* The following is a modification of [17, 4.3].

Lemma 4.3. *For each cardinal $\kappa \leq |\mathfrak{P}|$, there exists a sub-family $\mathcal{G} \subset \mathfrak{P}$ with the properties that $|\mathcal{G}| \geq \kappa$ and that $|\{P \in \mathcal{G} :\ a \notin P\}| < \kappa$ for each $a \in \bigcup\{P :\ P \in \mathcal{G}\}$.*

Proof. For each $a \in A \cup \{\mathbf{e}_A\}$, let $\mathfrak{P}_a$ be the set of prime ideals of the form $I{:}ab$ for some $b \in A$. We *claim* that there exists $a_0 \in A \cup \{\mathbf{e}_A\}$ such that $|\mathfrak{P}_{a_0}| \geq \kappa$ and such that, for each $a \in A$, either $|\mathfrak{P}_{a_0a}| < \kappa$ or $I{:}a_0a = I{:}a_0$. Indeed, assume the contrary. Then, since $|\mathfrak{P}_{\mathbf{e}_A}| \geq \kappa$, by induction, there exists a sequence $(a_n) \subset A$ such that $|\mathfrak{P}_{a_1\cdots a_n}| \geq \kappa$ and such that

$$I{:}a_1\cdots a_n \subsetneq I{:}a_1\cdots a_{n+1} \qquad (n \in \mathbb{N}).$$

This contradicts the definition of an continuity ideal. Hence the claim holds.

Put $\mathcal{G} = \mathfrak{P}_{a_0}$; this obviously satisfies $|\mathcal{G}| \geq \kappa$. Suppose that $a \in A$ and that $\mathcal{G}' = \{P \in \mathcal{G} : a \notin P\}$ has cardinality at least κ. Then, for each $P \in \mathcal{G}'$, because $a \notin P$ we have $P{:}a = P$. Thus $\mathcal{G}' \subset \mathfrak{P}_{a_0a}$, and hence $|\mathfrak{P}_{a_0a}| \geq \kappa$. Therefore, by the claim, we must have $I{:}a_0a = I{:}a_0$. We now show that $\mathcal{G}' = \mathcal{G}$. Assume towards a contradiction that $\mathcal{G}' \neq \mathcal{G}$, and let $P \in \mathcal{G} \setminus \mathcal{G}'$, say $P = I{:}a_0a_1$ for some $a_1 \in A$. Then, since $a \in P$ we have $a_1 \in I{:}a_0a = I{:}a_0$, so that $a_0a_1 \in I$. This implies that $P = A$, a contradiction. This proves that $\mathcal{G}$ has the desired property. □

Lemma 4.4. *$\sqrt{I}$ is the intersection of the prime ideals in $\mathfrak{P}$.*

Proof. This is based on the commutative prime kernel theorem due to Sinclair (see [5, Theorem 5.3.15] or [19, Theorem 11.4]). The proof of [17, Lemma 4.1] works almost verbatim. □

Lemma 4.5. *Every element in $\mathfrak{P}$ contains a minimal element.*

Proof. Assume toward a contradiction that there exists $(P_n = I{:}f_n) \subset \mathfrak{P}$ such that

$$P_1 \supsetneq P_2 \supsetneq \cdots \supsetneq P_n \supsetneq \cdots .$$

For each n, choose $a_n \in A$ such that $a_n \in P_n \setminus P_{n+1}$. Then we see that $a_1 \cdots a_n f_n \in I$ but $a_1 \cdots a_n f_{n+1} \notin I$. Thus

$$I{:}a_1 a_2 \cdots a_n \subsetneq I{:}a_1 a_2 \cdots a_{n+1} \qquad (n \in \mathbb{N});$$

a contradiction to I being an abstract continuity ideal. □

Lemma 4.6. *Let P be in $\mathfrak{P}$. Then there exists $a \notin P$ but $a \in Q$ for all $Q \in \mathfrak{P}$ such that $Q \not\subset P$.*

Proof. Assume the contrary. Pick $a_1 \notin P$. Suppose that we have already picked $a_1, \ldots, a_n \notin P$. By the assumption, we can find $Q_n \in \mathfrak{P}$ such that $a_1 \ldots a_n \notin Q_n$ and $Q_n \not\subset P$. We can then choose $a_{n+1} \in Q_n \setminus P$. The induction can be continued. We see that (a_n), (Q_n) constructed satisfy $a_1 \ldots a_n \notin Q_n$ but $a_1 \ldots a_{n+1} \in Q_n$ $(n \in \mathbb{N})$. Let $Q_n = I{:}f_n$. Then we see that $f_n \in I{:}a_1 a_2 \cdots a_{n+1} \setminus I{:}a_1 a_2 \cdots a_n$; a contradiction to I being an abstract continuity ideal. □

Lemma 4.7. *Let $\{P_\alpha :\ \alpha \in S\}$ be a subfamily of $\mathfrak{P}$. Then $\bigcap_{\alpha \in S} P_\alpha$ is also an abstract continuity ideal.*

Proof. Assume the contrary. Then there exists (a_n) such that

$$\big(\textstyle\bigcap_{\alpha \in S} P_\alpha\big){:}a_1 a_2 \cdots a_n \subsetneq \big(\textstyle\bigcap_{\alpha \in S} P_\alpha\big){:}a_1 a_2 \cdots a_{n+1} \qquad (n \in \mathbb{N}).$$

For each n, choose $b_n \in A$ such that $a_1 \cdots a_n b_n \notin \bigcap_{\alpha \in S} P_\alpha$ but $a_1 \cdots a_{n+1} b_n \in \bigcap_{\alpha \in S} P_\alpha$. Then choose $\alpha_n \in S$ such that $a_1 \cdots a_n b_n \notin P_{\alpha_n}$. We have $P_{\alpha_n} = I{:}f_{\alpha_n}$ for some $f_{\alpha_n} \in A$. We see that $a_1 \cdots a_n b_n f_{\alpha_n} \notin I$ but $a_1 \cdots a_{n+1} b_n f_{\alpha_n} \in I$. Thus

$$I{:}a_1 a_2 \cdots a_n \subsetneq I{:}a_1 a_2 \cdots a_{n+1} \qquad (n \in \mathbb{N});$$

a contradiction to I being an abstract continuity ideal. □

Lemma 4.8. *Let J be a semiprime ideal in A. Let $a, b \in A$ be such that $J{:}a$ and $J{:}b$ are prime ideals. Then the following are equivalent:*

(a) $J{:}a \subset J{:}b$,
(b) $ab \notin J$,
(c) $J{:}a = J{:}b$.

Proof. (a)⇒(b): Since J is semiprime and $J{:}b$ is a proper ideal in A, we see that $b \notin J{:}b$. So $b \notin J{:}a$, and therefore $ab \notin J$.

(b)⇒(c): Condition (b) implies that $a \notin J{:}b$. Let $f \in J{:}a$. Then $fa \in J \subset J{:}b$, and so, by the primeness of $J{:}b$, $f \in J{:}b$. Thus $J{:}a \subset J{:}b$. Similarly, we have $J{:}b \subset J{:}a$. □

Remark. In the case where I is semiprime, the above lemma shows that, for each $P = I{:}a \in \mathfrak{P}$, P is minimal in $\mathfrak{P}$ and $a \notin P$ but $a \in Q$ whenever $Q \in \mathfrak{P} \setminus \{P\}$.

We can now state the main result of this section.

Theorem 4.9. *Let I be an abstract continuity ideal of a commutative algebra A. Denote by $\mathfrak{P}_0$ the set of minimal ideals among the prime ideals of the form $I{:}a$ for some $a \in A$. Then:*

(i) $\sqrt{I} = \bigcap \{P : P \in \mathfrak{P}_0\}$;
(ii) $\mathfrak{P}_0$ *is a relatively compact family of prime ideals.*

Proof. The first assertion follows from Lemmas 4.4 and 4.5.

For the second one, let $(P_n) \subset \mathfrak{P}_0$. Set $J = \bigcap_{n=1}^{\infty} P_n$. By Lemma 4.6, there exists $a_n \in \bigcap_{i \neq n} P_i \setminus P_n$, and so $P_n = J{:}a_n$. Let $a \in A$ be such that $J{:}a$ is a prime ideal. We *claim* that $J{:}a \in \{P_n\}$. Indeed, we see that $a \notin J$, and thus $a \notin P_{n_0}$ for some n_0. So, $aa_{n_0} \notin J$. By Lemma 4.8, we deduce that $J{:}a = P_{n_0}$.

It then follows from Lemmas 4.7 and 4.3 (applied to J) that (P_n) must have a pseudo-finite subsequence. □

Corollary 4.10. *Let $\theta : A \to B$ be a homomorphism from a commutative Banach algebra A into a Banach algebra B. Then $\sqrt{\mathcal{I}(\theta)}$ is the intersection of a relatively compact family of prime ideals of the form $\mathcal{I}(\theta){:}a$ for $a \in A$.* □

Lemma 4.11. *Let I be an abstract continuity ideal of $\mathcal{C}_0(\Omega)$ for a locally compact space Ω. Then I is either a semiprime ideal or the whole of $\mathcal{C}_0(\Omega)$.*

Proof. The proof is the same as the proof that the continuity ideal of a discontinuous homomorphism from $\mathcal{C}_0(\Omega)$ into a Banach algebra is semiprime ([8],[18], cf. [5, Theorem 5.4.31]). □

Corollary 4.12. *Let Ω be a locally compact space.*

(i) *Let I be an abstract continuity ideal in $\mathcal{C}_0(\Omega)$. Denote by $\mathfrak{P}$ the set of prime ideals of the form $I{:}f$ for some $f \in \mathcal{C}_0(\Omega)$. Then:*
  (a) $I = \bigcap\{P : P \in \mathfrak{P}\}$*;*
  (b) $\mathfrak{P}$ *is a relatively compact family of prime ideals.*

(ii) *Conversely, let $\mathfrak{P}$ be a relatively compact family of prime ideals in $\mathcal{C}_0(\Omega)$. Then $\bigcap\{P : P \in \mathfrak{P}\}$ is an abstract continuity ideal in $\mathcal{C}_0(\Omega)$.* □

Corollary 4.13. *Let Ω be a locally compact space. Then each homomorphism from $\mathcal{C}_0(\Omega)$ into a Banach algebra is continuous on the intersection of a relatively compact family of prime ideals of the form $\mathcal{I}(\theta){:}f$ for $f \in \mathcal{C}_0(\Omega)$.* □

5. More properties of relatively compact families of prime ideals

In this section, let Ω be a locally compact space, and let $\mathfrak{P}$ be a non-empty relatively compact family of prime ideals in $\mathcal{C}_0(\Omega)$. Denote by $\mathfrak{Q}$ the collection of all the ideals that are unions of *countably* many ideals in $\mathfrak{P}$. We call $\mathfrak{Q}$ the *closure* of $\mathfrak{P}$; we shall show that it is indeed the smallest compact family of prime ideals containing $\mathfrak{P}$.

Note that an ideal in $\mathfrak{Q}$ is automatically prime in $\mathcal{C}_0(\Omega)$, and that the union of each pseudo-finite sequence of prime ideals in $\mathcal{C}_0(\Omega)$ is again a prime ideal in $\mathcal{C}_0(\Omega)$ (see the next lemma).

Lemma 5.1. *The union of finitely many prime ideals in $\mathcal{C}_0(\Omega)$ is either one of the given prime ideal or not an ideal. The union of countably many prime ideals in $\mathcal{C}_0(\Omega)$ is not equal $\mathcal{C}_0(\Omega)$.*

Proof. We prove the second clause only; the proof of the first one is similar. Let P_n ($n \in \mathbb{N}$) be prime ideals in $\mathcal{C}_0(\Omega)$. Choose $f_n \in \mathcal{C}_0(\Omega) \setminus P_n$. We can assume that $0 \le f_n \le 2^{-n}$. Set $f = \sum_{n=1}^{\infty} f_n$. Then $f \in \mathcal{C}_0(\Omega)$ but $f \notin P_n$ ($n \in \mathbb{N}$) since $f \ge f_n$. □

Lemma 5.2. *Each chain in $\mathfrak{Q}$ is well-ordered with respect to the inclusion; that is, each non-empty chain in $\mathfrak{Q}$ has a smallest element.*

Proof. Assume the contrary, then we can find an infinite chain $\cdots \subsetneq Q_n \subsetneq \cdots \subsetneq Q_1$ in $\mathfrak{Q}$. For each n, choose $P_n \in \mathfrak{P}$ such that $P_n \subset Q_n$ but $P_n \not\subset Q_{n+1}$. By the relative compactness of $\mathfrak{P}$ and without loss of generality, we can assume that $(P_n : n \in \mathbb{N})$ is a pseudo-finite sequence. Set $Q = \bigcup_{n=1}^{\infty} P_n$. Then $Q \in \mathfrak{Q}$, and for each $n \in \mathbb{N}$, either $Q_n \subset Q$ or $Q \subset Q_n$. Since $P_{n-1} \not\subset Q_n$, we must have $Q_n \subset Q$ $(n \geq 2)$. Choose $a \in Q_2 \setminus Q_3$. Then $a \notin Q_n$, and so $a \notin P_n$ $(n \geq 3)$. However, $a \in Q = \bigcup_{n=1}^{\infty} P_n$. This contradicts the pseudo-finiteness of (P_n). □

Lemma 5.3. *$\mathfrak{Q}$ is compact.*

Proof. Let (Q_n) be a sequence in $\mathfrak{Q}$. Let $P_n \in \mathfrak{P}$ such that $P_n \subset Q_n$. We can find a pseudo-finite subsequence (P_{n_i}); the union of which is denoted by Q. We have either $Q_{n_i} \subset Q$ or $Q \subset Q_{n_i}$. If there are infinitely many Q_{n_i} contained in Q, then those Q_{n_i} form a pseudo-finite sequence. On the other hand, if there are infinitely many Q_{n_i} containing Q, then those Q_{n_i} form a chain, and the previous lemma enable us to find an increasing sequence of ideals. Thus $\mathfrak{Q}$ is relatively compact. The result then follows from the definition of $\mathfrak{Q}$. □

Lemma 5.4. *$\mathfrak{Q}$ is the set of unions of pseudo-finite sequences of ideals in $\mathfrak{P}$.*

Proof. We need only to prove that each ideal $Q \in \mathfrak{Q}$ is the union of a pseudo-finite sequence in $\mathfrak{P}$. For this purpose, we only need to consider the case where $\mathfrak{P}$ is countable and that Q is the largest ideal in $\mathfrak{Q}$. It is easy to see that, in this case, any chain in $\mathfrak{Q}$ is countable.

Case 1: Q is the union of a chain of ideals in $\mathfrak{Q} \setminus \{Q\}$. By the countability and well-ordering of the chain, there exist $Q_1 \subsetneq Q_2 \subsetneq \cdots \subsetneq Q$ such that $Q = \bigcup_{n=1}^{\infty} Q_n$. For each n, choose $P_n \in \mathfrak{P}$ such that $P_n \subset Q_{n+1}$ but $P_n \not\subset Q_n$. By the relative compactness of $\mathfrak{P}$, we can choose a pseudo-finite subsequence (P_{n_i}) with union Q'. Then $Q' \subset Q$, and for each $i \geq 2$, either $Q_{n_i} \subset Q'$ or $Q' \subset Q_{n_i}$. Since $P_{n_i} \not\subset Q_{n_i}$, we must have $Q_{n_i} \subset Q'$ $(i \geq 2)$. Thus $Q = Q'$.

Case 2: Q is not the union of any chain of ideals in $\mathfrak{Q} \setminus \{Q\}$. Then any $P \in \mathfrak{Q} \setminus \{Q\}$ is contained in a maximal element of $\mathfrak{Q} \setminus \{Q\}$. Since Q cannot be the union of any finite number of prime ideals properly contained in Q, either $\mathfrak{Q} = \{Q\}$ which implies that $Q \in \mathfrak{P}$ or there exists infinitely many maximal elements of $\mathfrak{Q} \setminus \{Q\}$. In the latter case, let Q_n $(n \in \mathbb{N})$ be distinct maximal elements of $\mathfrak{Q} \setminus \{Q\}$. Choose $P_n \in \mathfrak{P}$ such that $P_n \subset Q_n$. By the relative compactness of $\mathfrak{P}$ and without loss of generality, we can assume that $(P_n : n \in \mathbb{N})$ is a pseudo-finite sequence. Set $Q' = \bigcup_{n=1}^{\infty} P_n$. Then $Q' \in \mathfrak{Q}$, $Q' \subset Q$, and for each $n \in \mathbb{N}$, either $Q_n \subset Q'$ or $Q' \subset Q_n$. The maximality and distinction of Q_n's imply that $Q' \subset Q_n$ for at most one $n \in \mathbb{N}$. The maximality of Q_n's again implies that $Q' = Q$.

In both cases, we see that Q is the union of a pseudo-finite sequence in $\mathfrak{P}$. □

Summary; note that the property (iii) is indeed a consequence of (ii):

Proposition 5.5. *$\mathfrak{Q}$ satisfies the following:*

(i) *$\mathfrak{Q}$ is the set of unions of pseudo-finite sequences of ideals in $\mathfrak{P}$;*
(ii) *$\mathfrak{Q}$ is compact;*
(iii) *each chain in $\mathfrak{Q}$ is well-ordered;*
(iv) $\bigcap \mathfrak{P} = \bigcap \mathfrak{Q}$. □

From (i), we see that $\mathfrak{Q}$ is the smallest compact family of prime ideals containing $\mathfrak{P}$. Property (iii) also shows that the intersection of $\mathfrak{P}$ is equal the intersection of its minimal elements.

In the remaining of the section, we consider $\mathfrak{Q}$ to be any compact family of prime ideals in $\mathcal{C}_0(\Omega)$.

Lemma 5.6. *Let P be in $\mathfrak{Q}$. Then there exists $a \notin P$ but $a \in Q$ for all $Q \in \mathfrak{Q}$ such that $Q \not\subset P$.*

Proof. Assume the contrary. As in Lemma 4.6, we can construct $(a_n) \subset A$, $(Q_n) \subset \mathfrak{Q}$ satisfying $a_1 \ldots a_n \notin Q_n$ but $a_1 \ldots a_{n+1} \in Q_n$ ($n \in \mathbb{N}$). By the compactness, (Q_n) has a pseudo-finite subsequence (Q_{n_i}). However, $a_1 \ldots a_{n_2} \in Q_{n_1}$ but $a_1 \ldots a_{n_2} \notin Q_{n_i}$ ($i \geq 2$); a contradiction to the pseudo-finiteness. $\square$

We say that an ideal Q is a *roof* of $\mathfrak{Q}$ if it is the union of the ideals in a maximal chain in $\mathfrak{Q}$. A roof must be either a prime ideal in $\mathcal{C}_0(\Omega)$ or $\mathcal{C}_0(\Omega)$ itself.

Lemma 5.7. *$\mathfrak{Q}$ has only finitely many roofs. Also, there are only finite many maximal modular ideals in $\mathcal{C}_0(\Omega)$ such that each of them contains an ideal in $\mathfrak{Q}$.*

Proof. We shall prove that there can be only finitely many *disjoint* maximal chains in $\mathfrak{Q} \setminus \{\mathcal{C}_0(\Omega)\}$; the lemma then follows. Assume the contrary that $\mathfrak{C}_n$ ($n \in \mathbb{N}$) are disjoint maximal chains in $\mathfrak{Q} \setminus \{\mathcal{C}_0(\Omega)\}$. Pick $Q_n \in \mathfrak{C}_n$. Without loss of generality, we can suppose that (Q_n) is pseudo-finite; the union is denoted by Q. We see that $Q \in \mathfrak{Q} \setminus \{\mathcal{C}_0(\Omega)\}$, and so $Q \in \mathfrak{C}_n$ ($n \in \mathbb{N}$), contradicting the disjointness of $\mathfrak{C}_n$'s. $\square$

The following lemma and proposition are based on a suggestion of an anonymous referee of an initial version of our previous paper.

Lemma 5.8. *Suppose that $\mathfrak{Q}$ is a compact family of prime ideals in $\mathcal{C}_0(\Omega)$ with only one maximal element Q. Set $I = \bigcap \mathfrak{Q}$. Let $a \in \mathcal{C}_0(\Omega) \setminus Q$ and let $b \in Q$. Then there exists $s \in Q$ such that $as - b \in I$.*

Proof. It is standard that for each prime ideal $P \subset Q$ there exists $s \in Q$ such that $as - b \in P$.

Assume toward a contradiction that for all $s \in Q$ we have $as - b \notin I$. Set $s_1 = 0$, $b_1 = b - as_1 = b$, and $\mathfrak{Q}_1 = \{P \in \mathfrak{Q} : \ b_1 \notin P\}$. Suppose that we have already construct $s_n \in Q$, $b_n = b - as_n$, and $\mathfrak{Q}_n = \{P \in \mathfrak{Q} : \ b_n \notin P\}$ such that

$$\mathfrak{Q}_n \subsetneq \cdots \subsetneq \mathfrak{Q}_1.$$

By the assumption, we have $\mathfrak{Q}_n \neq \emptyset$. Choose $P_0 \in \mathfrak{Q}_n$. Since $\dfrac{b_n}{\sqrt{|b_n|}} \in Q$. There exists $s' \in Q$ such that $as' - \dfrac{b_n}{\sqrt{|b_n|}} \in P_0$. Set $s_{n+1} = s_n + s'\sqrt{|b_n|}$,

$$b_{n+1} = b - as_{n+1} = \left(\frac{b_n}{\sqrt{|b_n|}} - as' \right) \sqrt{|b_n|},$$

and $\mathfrak{Q}_{n+1} = \{P \in \mathfrak{Q} : \ b_{n+1} \notin P\}$. We see that $\mathfrak{Q}_{n+1} \subsetneq \mathfrak{Q}_n$. Thus the construction can be continued inductively.

Choose $P_n \in \mathfrak{Q}_n \setminus \mathfrak{Q}_{n+1}$ for each n; then $b_m \in P_n$ ($m > n$) but $b_m \notin P_n$ ($m \leq n$). The compactness implies that there exists a pseudo-finite subsequence (P_{n_i}), whose union is denoted by P. For all $j > 1$, $b_{n_j} \in P_{n_1}$ so, $b_{n_j} \in P$. In particular, $b_{n_2} \in P$.

On the other hand, for all $2 \le i$, $b_{n_2} \notin P_{n_i}$, and so $b_{n_2} \notin P = \bigcup_{i=2}^{\infty} P_{n_i}$; a contradiction. $\square$

Proposition 5.9. *Suppose that $\mathfrak{Q}$ is a compact family of prime ideals in $\mathcal{C}_0(\Omega)$ with only one maximal element Q. Set $I = \bigcap \mathfrak{Q}$. Let $P \in \mathfrak{Q}$, and let A be a subalgebra of $\mathcal{C}_0(\Omega)$. Suppose that $\mathcal{C}_0(\Omega) = A + P$ and $A \cap P$ is the intersection of a sub-family of $\mathfrak{Q}$. Let B be a subalgebra of A such that B is maximal with respect to the property that $B \cap Q \subset I$. Then $\mathcal{C}_0(\Omega) = B + Q$.*

Proof. Note that $Q/I \subset \operatorname{rad} \mathcal{C}_0(\Omega)/I$; this follows from the previous lemma. Also that $I \subset B$, so indeed $B \cap Q = I$.

Claim: for each $a \in A \setminus Q$ and each $b \in A \cap Q$, there exists $s \in A \cap Q$ such that $as - b \in I$. Indeed, by the previous lemma, there exists $s \in Q$ such that $as - b \in I$. Write $s = c + p$ where $c \in A$ and $p \in P$. Then $ap + (ac - b) \in I \subset A$ implies that $ap \in A \cap P$. Since $a \notin Q$ and $A \cap P$ is the intersection of a family of prime ideals contained in Q, we must have $p \in A \cap P$. Thus $s = c + p \in A \cap Q$.

The above claim shows in particular that $(A \cap Q)/I \subset \operatorname{rad} A/I$. We shall prove that $A/I = B/I \oplus (A \cap Q)/I$; the proposition then follows.

Assume toward a contradiction. Let $a \in A/I$ but $a \notin B/I \oplus (A \cap Q)/I$. By the maximality of B, there exists a non-zero polynomial $q(X)$ with coefficients in B/I such that $q(a) \in Q/I$. Let $q(X)$ be such a polynomial with smallest degree. Then $q'(a) \notin Q/I$, where $q'(X)$ is the formal derivative. Let $s \in \mathcal{C}_0(\Omega)/I$. Then

$$q(a + q'(a)s) = q(a) + q'(a)^2 s + \ldots + q'(a)^n q^{(n)}(a) \frac{s^n}{n!};$$

where $q^{(k)}(X)$ is the formal k^{th} derivative of $q(X)$. By the claim, there exists $d \in (A \cap Q)/I$ such that $q(a) = q'(a)^2 d$. So

$$q(a + q'(a)s) = q'(a)^2 \left(d + s + \ldots + q'(a)^{n-2} q^{(n)}(a) \frac{s^n}{n!} \right).$$

Since $(A/(A \cap P))^{\#} \cong (\mathcal{C}_0(\Omega)/P)^{\#}$ is Henselian, there exist $s \in \operatorname{rad} A/I$ such that

$$d + s + \ldots + q'(a)^{n-2} q^{(n)}(a) \frac{s^n}{n!} \in (A \cap P)/I;$$

([5], Theorem 2.4.30 and Proposition 1.6.3). It follows that $s \in (A \cap Q)/I$ and

$$q(a + q'(a)s) \in (A \cap P)/I.$$

If we set $b = a + q'(a)s$, then $b \in A/I$ but $b \notin B/I \oplus (A \cap Q)/I$, $q(b) \in (A \cap P)/I$, and $q(X)$ is the smallest degree non-zero polynomial with coefficients in B/I such that $q(b) \in Q/I$. So without loss of generality, we can assume that $q(a) \in (A \cap P)/I$.

The case where $q(a) \in (A \cap P)/I$: Then $d \in (A \cap P)/I$. Since $\mathcal{C}_0(\Omega)^{\#}/I$ is Henselian, there exist $t \in \operatorname{rad} \mathcal{C}_0(\Omega)/I$ such that

$$0 = d + t + \ldots + q'(a)^{n-2} q^{(n)}(a) \frac{t^n}{n!}.$$

Since $(A \cap P)/I$ is an ideal in $\mathcal{C}_0(\Omega)/I$, it follows that $t \in (A \cap P)/I$. Set $c = a + q'(a)t$. Then $c \in A/I$, $q(c) = 0$, and $q(X)$ is a non-zero polynomial with coefficients in B/I with the smallest degree such that $q(c) \in Q/I$. Let $p(X)$ be any polynomial with coefficients in B/I such that $p(c) \in Q/I$. We see that there exist non-zero element $u \in B/I$ and a polynomial $h(X)$ with coefficient in B/I such that $up(X) = q(X)h(X)$. Then $up(c) = q(c)h(c) = 0$, and since $u \notin Q/I$,

we deduce that $p(c) = 0$. The maximality of B then implies that $c \in B/I$, and so $a = c - q'(a)t \in B/I + Q/I$; a contradiction. $\square$

A special case of the previous proposition is when $A = \mathcal{C}_0(\Omega)$.

6. The main results

In this section, we shall show the connection between continuity ideals (as well as kernels) of homomorphisms from $\mathcal{C}_0(\Omega)$ into Banach algebras and intersections of (relatively) compact families of prime ideals. One direction is an immediate consequence of the results in section 4, so most of this section concerns the converse.

We shall need some basic complex algebraic-geometry results. Our references for algebraic geometry will be [16]. For a set $S \subset \mathbb{C}[Z_1, Z_2, \dots, Z_n]$, denote by $\mathcal{V}(S)$ the *variety* (i.e., common zero set) of S in $\mathbb{C}^n$. For each prime ideal Q in $\mathbb{C}[Z_1, \dots, Z_n]$, the variety $\mathcal{V}(Q)$ is *irreducible*. The topology considered on complex spaces will be the Euclidean topology. We shall need the fact that, for each irreducible variety V and each variety W not containing V, $V \setminus W$ is dense and (relatively) open in V [16, Chapter IV, Theorem 2.11].

Notation. For clarity, we shall use X_i, Y_j for variables, x_i, y_j for complex numbers, and a_i, b_j for elements of an algebra. When there is no ambiguity, we shall use boldface characters to denote tuples of elements of the same type; for example, we set

$$\boldsymbol{X} = (X_1, X_2, \dots, X_m) \quad \text{or} \quad \boldsymbol{y} = (y_1, \dots, y_n)\,.$$

In the case where $\boldsymbol{X} = (X_1, \dots, X_m)$, we also denote by $\mathbb{C}_{\boldsymbol{X}}$ the corresponding space $\mathbb{C}^m$.

Lemma 6.1. *Let $m, n \in \mathbb{N}$, and let Q be a prime ideal in $\mathbb{C}[\boldsymbol{X}, \boldsymbol{Y}]$, where $\boldsymbol{X} = (X_1, \dots, X_m)$ and $\boldsymbol{Y} = (Y_1, \dots, Y_n)$. Consider $Q_{\boldsymbol{X}} = Q \cap \mathbb{C}[\boldsymbol{X}]$ as a prime ideal in $\mathbb{C}[\boldsymbol{X}]$. Let V be the variety of Q, and let $V_{\boldsymbol{X}}$ be the variety of $Q_{\boldsymbol{X}}$. Let π be the natural projection $\mathbb{C}_{\boldsymbol{X},\boldsymbol{Y}} \to \mathbb{C}_{\boldsymbol{X}}$. Then $\pi : V \to V_{\boldsymbol{X}}$ and there exists a dense open subset U of V such that $\pi : U \to V_{\boldsymbol{X}}$ is an open map.*

Proof. Obviously, $\pi : V \to V_{\boldsymbol{X}}$. Without loss of generality, let $(X_1, \dots, X_k)$ be a transcendental basis for $\mathbb{C}[\boldsymbol{X}]$ modulo $Q_{\boldsymbol{X}}$. We consider $\mathbb{C}^k = \mathbb{C}_{X_1,\dots,X_k}$. Denote by π_1 the natural projection $\mathbb{C}_{\boldsymbol{X},\boldsymbol{Y}} \to \mathbb{C}^k$, and by π_2 the natural projection $\mathbb{C}_{\boldsymbol{X}} \to \mathbb{C}^k$.

By [17, Lemma 6.3], there exist dense open subsets U and $U_{\boldsymbol{X}}$ of V and $V_{\boldsymbol{X}}$, respectively, such that $\pi_1 : U \to \mathbb{C}^k$ and $\pi_2 : U_{\boldsymbol{X}} \to \mathbb{C}^k$ are open maps. Inspecting the proof of [17, Lemma 6.3], we see that $U_{\boldsymbol{X}}$ can be chosen as $V_{\boldsymbol{X}} \setminus V_0$, where V_0 is a proper subvariety of $V_{\boldsymbol{X}}$, and that π_2 is even a local homeomorphism from $U_{\boldsymbol{X}}$ onto an open subset of $\mathbb{C}^k$. Since V_0 has dimension at most $k-1$ [16, Chapter IV], we can further require that

$$\pi_2(U_{\boldsymbol{X}}) \cap \pi_2(V_{\boldsymbol{X}} \setminus U_{\boldsymbol{X}}) = \emptyset.$$

Let $W = \pi_1(U) \cap \pi_2(U_{\boldsymbol{X}})$. Then W is an open set in $\mathbb{C}^k$. It can be seen that W is dense in $\pi_1(U)$. Set

$$U' = U \cap \pi_1^{-1}(W).$$

Then U' is a dense open subset of V. Let $(\boldsymbol{x}, \boldsymbol{y}) \in U'$. We can see that $\pi : U' \to V_{\boldsymbol{X}}$ is an open map. $\square$

Proposition 6.2. *Let $A = \mathcal{C}_0(\Omega)$ for a locally compact space Ω, and let $\mathfrak{Q}$ be a non-empty compact family of non-modular prime ideals in A. Suppose that each chain in $\mathfrak{Q}$ is countable. Then there exist a cardinal κ, a free ultrafilter $\mathcal{U}$ on κ, and, for each $P \in \mathfrak{P}$, a homomorphism $\theta_P : A \to (\mathbb{C}^\kappa/\mathcal{U})^\circ$ such that:*

(a) $\ker\theta_P = P$ $(P \in \mathfrak{P})$, *and*
(b) *the set $\{\theta_P(a) : P \in \mathfrak{P}\}$ is finite for each $a \in A$.*

Proof. Note that A must be non-unital. Since each $P \in \mathfrak{Q}$ is a non-modular prime ideal in A, it is a prime ideal in $A^\#$. For each $Q \in \mathfrak{Q}$, set

$$\mathfrak{Q}_Q = \{P \in \mathfrak{Q} : \ P \subset Q\}\,,$$

and set $I_Q = \bigcap \mathfrak{Q}_Q$. We start the proof with some lemmas:

Lemma 6.3. *Let $Q_* \subset Q^* \in \mathfrak{Q}$. Let $A_* \supset A^*$ be subalgebras of A such that $A^* \cap Q^* \subset I_{Q^*}$, and $A_* \cap Q_* = I_{Q_*}$. Suppose further that $A_* + Q_* = A$. Let $\mathfrak{C}$ be a chain in $\mathfrak{Q}_{Q^*}$ where each ideal in $\mathfrak{C}$ contains Q_*. Then, for each $Q \in \mathfrak{C}$, we can find a subalgebra $A_Q \subset A$, such that the following conditions is satisfied:*

(i) $A_Q \cap Q = I_Q$ *and* $A = A_Q + Q$ $(Q \in \mathfrak{C})$;
(ii) *for each $Q_1 \subset Q_2 \in \mathfrak{C}$, we have $A_* \supset A_{Q_1} \supset A_{Q_2} \supset A^*$.*

Proof. Assume toward a contradiction that the lemma fails. Let Q_*, Q^*, A_*, A^*, and $\mathfrak{C}$ be as above and such that the lemma fails and that $\mathrm{o}(\mathfrak{C})$ is smallest. Proposition 5.9 implies that $\mathfrak{C}$ must be infinite.

If $\mathfrak{C}$ is order isomorphic to ω the first infinite ordinal; say

$$\mathfrak{C} = \{Q_1 \subset Q_2 \subset \cdots\}\,.$$

Then, Proposition 5.9 enable us to construct (A_{Q_n}) inductively.

In general, we index $\mathfrak{C}$ increasingly by $\alpha \in \mathrm{o}(\mathfrak{C})$, so that $\mathfrak{C} = (Q_\alpha)$. There exists a sequence (α_n) in $\mathrm{o}(\mathfrak{C})$ converging in the order topology to $\gamma = \sup \mathrm{o}(\mathfrak{C})$. If $\gamma \in \mathrm{o}(\mathfrak{C})$, we can, by Proposition 5.9 construct A_{Q_γ} first. As in the previous paragraph, we can (then) find $A_{Q_{\alpha_n}}$ $(n \in \mathbb{N})$ satisfying both the conditions (i) and (ii). The α_n's divide $\mathrm{o}(\mathfrak{C}) \setminus \{\gamma\}$ into chains which are order isomorphic to ordinals strictly smaller than $\mathrm{o}(\mathfrak{C})$. The minimality of $\mathrm{o}(\mathfrak{C})$ then allows us to find A_Q $(Q \in \mathfrak{C})$ satisfying the conditions (i) and (ii).

Thus, in any case, we have a contradiction. □

Lemma 6.4. *For each $Q \in \mathfrak{Q}$, we can find a subalgebra $A_Q \subset A$ such that the following conditions is satisfied:*

(i) $A_Q \cap Q = I_Q$ *and* $A = A_Q + Q$ $(Q \in \mathfrak{Q})$;
(ii) *for each $Q_1 \subset Q_2 \in \mathfrak{Q}$, we have $A_{Q_1} \supset A_{Q_2}$.*

Proof. This follows from Zorn's lemma, the fact that all prime ideals containing a given prime ideal form a chain, and the previous lemma. □

Let κ be the set of all tuples of the form $(\delta; \mathfrak{G}; a_1, \ldots, a_m)$, where $\delta > 0$, $\mathfrak{G}$ is a non-empty finite subsets of $\mathfrak{Q}$, and $(a_1, \ldots, a_m)$ is a non-empty finite sequence of distinct elements in A. Define a partial order $\prec$ on κ by setting

$$(\delta; \mathfrak{G}; a_1, a_2, \ldots, a_m) \prec (\delta'; \mathfrak{G}'; a'_1, a'_2, \ldots, a'_{m'})$$

if $\delta > \delta'$, $\mathfrak{G} \subset \mathfrak{G}'$, $(a_1, \ldots, a_m)$ is a subsequence of $(a'_1, a'_2, \ldots, a'_{m'})$. Then $(\kappa, \prec)$ is a net. Fix an ultrafilter $\mathcal{U}$ on κ majorizing this net.

Lemma 6.5. *For each $w = (\delta; \mathfrak{G}; a_1, \ldots, a_m) \in \kappa$. Then we can find, for each $P \in \mathfrak{G}$, a finite sequence of complex numbers*

$$\tau_P(w) = \boldsymbol{x}^{(P)} = (x_1^{(P)}, \ldots, x_m^{(P)})$$

satisfying all the following conditions:

(i) $p(x_1^{(P)}, \ldots, x_m^{(P)}) = 0$ *for each* $p \in \mathbb{C}[X_1, \ldots, X_m]$ *with* $p(a_1, \ldots, a_m) \in P$;
(ii) *for each* $1 \le k \le m$ *with* $a_k \notin P$, *we have* $x_k^{(P)} \neq 0$;
(iii) $|x_k^{(P)}| \le \delta \quad (1 \le k \le m)$.
(iv) *for each* $1 \le k \le m$ *and each* $P \subset Q \in \mathfrak{G}$ *such that* $a_k \in A_Q$, *we have* $x_k^{(P)} = x_k^{(Q)}$.

Proof. For each $P \in \mathfrak{G}$, set $\boldsymbol{a}^P = (a_1, \ldots, a_m) \cap A_P$ and set $\boldsymbol{X}^P = (X_i : \ a_i \in A_P)$. Without loss of generality, we can assume that none of these is empty. Note that, when $P \subset Q \in \mathfrak{G}$ then $\boldsymbol{a}^Q \subset \boldsymbol{a}^P$ and $\boldsymbol{X}^Q \subset \boldsymbol{X}^P$.

Denote by π_P the projection from $\mathbb{C}_{\boldsymbol{X}}$ onto $\mathbb{C}_{\boldsymbol{X}^P}$, and π_{PQ} the projection from $\mathbb{C}_{\boldsymbol{X}^P}$ onto $\mathbb{C}_{\boldsymbol{X}^Q}$ $(P \subset Q \in \mathfrak{G})$. We also conveniently consider $\mathbb{C}_{\boldsymbol{X}^P}$'s as subspaces of $\mathbb{C}_{\boldsymbol{X}}$.

For each $Q \in \mathfrak{G}$, define $\widetilde{Q} = \{p \in \mathbb{C}[\boldsymbol{X}] : \ p(\boldsymbol{a}) \in Q\}$, and

$$\widehat{Q} = \left\{ p \in \mathbb{C}[\boldsymbol{X}^Q] : \ p(\boldsymbol{a}^Q) \in Q \right\} .$$

We see that for $P \subset Q \in \mathfrak{G}$, $\widetilde{P} \subset \widetilde{Q}$ are prime ideals in $\mathbb{C}[\boldsymbol{X}]$, and $\widehat{Q} = \widetilde{Q} \cap \mathbb{C}[\boldsymbol{X}^Q]$ is a prime ideal in $\mathbb{C}[\boldsymbol{X}^Q]$. Also, since $A_Q \cap Q = A_Q \cap P$ for each $P \subset Q \in \mathfrak{G}$, we see that

$$\widehat{Q} = \left\{ p \in \mathbb{C}[\boldsymbol{X}^Q] : \ p(\boldsymbol{a}^Q) \in P \right\} = \widetilde{P} \cap \mathbb{C}[\boldsymbol{X}^Q];$$

and thus $\widehat{Q} = \mathbb{C}[\boldsymbol{X}^Q] \cap \widehat{P}$.

It follows from Lemma 6.1 that there exist, for each $Q \in \mathfrak{G}$, dense open subsets U_Q of $\mathcal{V}(\widetilde{Q})$ and W_Q of $\mathcal{V}(\widehat{Q})$, respectively, such that:

- $\pi_Q : U_Q \to W_Q$ is an open map;
- $\pi_{PQ} : W_P \to W_Q$ is an open map $(P \subset Q \in \mathfrak{G})$.

(We have only finitely many varieties here.)

For each $Q \in \mathfrak{G}$, set

$$V_Q = \bigcup_{1 \le r \le m, \ a_r \notin Q} \{\boldsymbol{x} = (x_1, \ldots, x_m) : \ x_r = 0\} .$$

Then V_Q is a variety which does not contain $\mathcal{V}(\widetilde{Q})$, and so $\mathcal{V}(\widetilde{Q}) \setminus V_Q$ is also a dense open subset of $\mathcal{V}(\widetilde{Q})$ because $\mathcal{V}(\widetilde{Q})$ is irreducible. Therefore, $U_Q \setminus V_Q$ is again a dense open subset of $\mathcal{V}(\widetilde{Q})$.

Set

$$\Delta = \{\boldsymbol{x} = (x_1, \ldots, x_m) : \ |x_r| < \delta \ (1 \le r \le m)\} .$$

Finally, set $U'_Q = (U_Q \setminus V_Q) \cap \Delta$ and $W'_Q = W_Q \cap \Delta$. Note that the origin $\mathbf{0}$ is in $\mathcal{V}(\widetilde{Q})$ and $\mathcal{V}(\widehat{Q})$ $(Q \in \mathfrak{G})$. So U'_Q (respectively, W'_Q) is a non-empty open subset of $\mathcal{V}(\widetilde{Q})$ (respectively, $\mathcal{V}(\widehat{Q})$).

Choosing any $\boldsymbol{x}^{(Q)} \in U'_Q$ will ensure the conditions (i), (ii), and (iii) are satisfied.

Fix $P \subset Q \in \mathfrak{G}$. Since $U_P \setminus V_P$ is dense in $\mathcal{V}(\widetilde{P})$ and $U'_Q \subset \mathcal{V}(\widetilde{Q}) \subset \mathcal{V}(\widetilde{P})$, we have $\pi_Q(U'_P) \cap \pi_Q(U'_Q)$ is dense (and open) in $\pi_Q(U'_Q)$ (which is open in W_Q). Note

that $\pi_Q(U'_P) = \pi_{PQ}[\pi_P(U'_P)]$. In fact, we see that for every dense open subset D of $\pi_P(U'_P)$, $\pi_{PQ}(D) \cap \pi_Q(U'_Q)$ is dense and open in $\pi_Q(U'_Q)$.

Thus, we can define dense open subsets D_Q of $\pi_Q(U'_Q)$ as follows: For P minimal in $\mathfrak{G}$, set $D_P = \pi_P(U'_P)$. Then, define inductively,

$$D_Q = \bigcap_{P\in\mathfrak{G},\ P\subsetneq Q} \pi_{PQ}(D_P) \cap \pi_Q(U'_Q).$$

The non-emptiness of $D'_P s$ allows us to choose $\boldsymbol{x}^{(Q)} \in U'_Q$ such that the condition (iv) is satisfied (inductively, starting from the maximal elements in $\mathfrak{G}$; note that the prime ideals containing a given prime ideal form a chain). □

For each $P \in \mathfrak{Q}$, define $\xi_P : A \to \mathbb{C}^\kappa$ as follows: For each $a \in A$ and each

$$w = (\delta; \mathfrak{G}; a_1, \dots, a_m) \in \kappa,$$

if $P \in \mathfrak{G}$ and if a is in $(a_1, \dots, a_m)$, say $a = a_k$ (there is at most one such k), then $\xi_P(a)(w) = \tau_P(w)(k)$; otherwise, $\xi_P(a)(w) = 0$. Define $\theta_P(a)$ to be the equivalence class in $\mathbb{C}^\kappa/\mathcal{U}$ containing $\xi_P(a)$.

Let $P \in \mathfrak{Q}$. By Lemma 6.5(i), the map θ_P is an algebra homomorphism and $P \subset \ker\theta_P$; by combining this with Lemma 6.5(ii), we see that $\ker\theta_P$ is exactly P. By Lemma 6.5(iii), the image of θ_P is contained in $(\mathbb{C}^\kappa/\mathcal{U})^\circ$. By Lemma 6.5(iv), for each $P \subset Q \in \mathfrak{Q}$ and each $a \in A_Q$, we have $\theta_P(a) = \theta_Q(a)$.

Let (P_n) be any pseudo-finite sequence in $\mathfrak{Q}$, and let $Q = \bigcup_{n=1}^\infty P_n$. Let $a \in A$. Write $a = b + x$, where $b \in A_Q$ and $x \in Q$. Then $x \in P_n$ for $n > n_0$, for some n_0. Therefore, for $n > n_0$, $\theta_{P_n}(a) = \theta_{P_n}(b) = \theta_Q(b)$. Thus, the sequence $(\theta_{P_n}(a))$ is eventually constant.

Finally, assume toward a contradiction that the set $\{\theta_P(a) : P \in \mathfrak{Q}\}$ is infinite for some $a \in A$. Let (P_n) be a sequence in $\mathfrak{Q}$ such that all $\theta_{P_n}(a)$ $(n \in \mathbb{N})$ are distinct. By the compactness of $\mathfrak{Q}$, (P_n) contains a pseudo-finite subsequence (P_{n_i}). However, the previous paragraph shows that the set $\left\{\theta_{P_{n_i}}(a) : i \in \mathbb{N}\right\}$ is finite. This is a contradiction.

This finished the proof of Proposition 6.2. □

Remark. The idea of the above proof originate from the approach in [7] of the theorem [11] of Esterle on embedding integral domains into radical Banach algebras.

We are now ready to prove our main results.

Theorem 6.6. *Let Ω be a locally compact space.*

(i) *Let θ be a homomorphism from $\mathcal{C}_0(\Omega)$ into a radical Banach algebra R. Then $\ker\theta$ is the intersection of a relatively compact family of non-modular prime ideals in $\mathcal{C}_0(\Omega)$.*

(ii) (CH) *Let I be the intersection of a relatively compact family of non-modular prime ideals in $\mathcal{C}_0(\Omega)$ such that I is also the intersection of a countable family of prime ideals. Suppose that*

$$\left|\mathcal{C}_0(\Omega)/I\right| = \mathfrak{c}\,.$$

Then there exists a homomorphism θ from $\mathcal{C}_0(\Omega)$ into a radical Banach algebra such that $\ker\theta = I$.

Proof. (i) Since θ maps into a radical algebra, we see that $\ker\theta{:}f$ is non-modular for each $f \in \mathcal{C}_0(\Omega)$. In this case, Theorem 1.1 shows that $\ker\theta = \mathcal{I}(\theta)$, and so, it is an abstract continuity in $\mathcal{C}_0(\Omega)$. The result will then follow from Corollary 4.12(i).

(ii) Let $\mathfrak{P}$ be a relatively compact family of non-modular prime ideals in $\mathcal{C}_0(\Omega)$ such that $I = \bigcap \mathfrak{P}$. We can assume that $\mathfrak{P} \neq \emptyset$. By keeping only the minimal elements in $\mathfrak{P}$, we can suppose that $P \not\subset Q$ $(P \neq Q \in \mathfrak{P})$. Since I is the intersection of a countable family of prime ideals, by Lemma 5.6, we see that $\mathfrak{P}$ is countable. Let $\mathfrak{Q}$ be the set of all the ideals that are unions of (countably many) prime ideals in $\mathfrak{P}$. It is easy to see that every chain in $\mathfrak{Q}$ is countable.

Let $\theta_P : \mathcal{C}_0(\Omega) \to (\mathbb{C}^\kappa/\mathcal{U})^\circ$ $(P \in \mathfrak{Q})$ be homomorphisms as in Proposition 6.2. Let B be the subalgebra of $(\mathbb{C}^\kappa/\mathcal{U})^\circ$ generated by all the images of θ_P $(P \in \mathfrak{Q})$. Then B is a non-unital integral domain. We also have

$$|B| = \left|\bigcup_{P\in\mathfrak{Q}} \theta_P(\mathcal{C}_0(\Omega))\right| = \left|\bigcup_{a\in\mathcal{C}_0(\Omega)} \{\theta_P(a) :\ P \in \mathfrak{Q}\}\right| = \mathfrak{c}\,;$$

since, for each $b \in a + I$, we have

$$\{\theta_P(b) :\ P \in \mathfrak{Q}\} = \{\theta_P(a) :\ P \in \mathfrak{Q}\}\,,$$

which is finite. Thus [11] there exists an embedding $\psi : B \to R_0$ where R_0 is a universal radical Banach algebra; for example, $R_0 = L^1(\mathbb{R}^+, \omega)$ for ω a radical weight bounded near the origin.

Consider the following map:

$$\theta : \mathcal{C}_0(\Omega) \to \prod_{P\in\mathfrak{Q}} R_0\,,\ a \mapsto \big((\psi \circ \theta_P)(a) :\ P \in \mathfrak{Q}\big)\,.$$

Then θ is a homomorphism with kernel $\bigcap \mathfrak{Q} = I$. We see, by Proposition 6.2, that the image of θ is in $\ell^\infty(\mathfrak{Q}, R_0)$, and indeed is in its radical R. Thus $\theta : \mathcal{C}_0(\Omega) \to R$ is the required homomorphism. □

Theorem 6.7. *Let Ω be a locally compact space.*

(i) *Let θ be a homomorphism from $\mathcal{C}_0(\Omega)$ into a Banach algebra B. Then $\mathcal{I}(\theta)$ is the intersection of a relatively compact family of prime ideals in $\mathcal{C}_0(\Omega)$.*

(ii) (CH) *Let I be the intersection of a relatively compact family of prime ideals in $\mathcal{C}_0(\Omega)$ such that I is also the intersection of a countable family of prime ideals. Suppose that*

$$\big|\mathcal{C}_0(\Omega)/I\big| = \mathfrak{c}\,.$$

Then there exists a homomorphism θ from $\mathcal{C}_0(\Omega)$ into a Banach algebra such that $\mathcal{I}(\theta) = I$.

Proof. (i) The continuity ideal $\mathcal{I}(\theta)$ is an abstract continuity ideal in $\mathcal{C}_0(\Omega)$. The result will follow from Corollary 4.12(i).

(ii) Let $\mathfrak{P}$ be a relatively compact family of prime ideals in $\mathcal{C}_0(\Omega)$ such that $I = \bigcap \mathfrak{P}$. We can assume that $\mathfrak{P} \neq \emptyset$. As in the previous proof, we can assume that $\mathfrak{P}$ is countable.

Denote by $\mathfrak{P}_0$ the set of non-modular ideals in $\mathfrak{P}$. Let $\mathfrak{Q}'$ be the collection of all the ideals that are unions of (countably many) prime ideals in $\mathfrak{P} \setminus \mathfrak{P}_0$. By Lemma 5.7, $\mathfrak{Q}'$ has only finitely many roofs. Denote by $Q_1, \ldots, Q_n$ the roofs of $\mathfrak{Q}'$, and set $\mathfrak{P}_i = \{P \in \mathfrak{P} \setminus \mathfrak{P}_0 :\ P \subset Q_i\}$.

Let $1 \le k \le n$. First, it is easy to see that Q_k is a modular prime ideal. Pick a modular identity u for Q_k, and pick $a \notin Q_k$. Then $a - au \in Q_k$, and so, by Lemma 5.8, there exists $v \in Q_k$ such that $a - au - av \in \bigcap \mathfrak{P}_k$. It follows easily that $u + v$ is a modular identity for $\bigcap \mathfrak{P}_k$; denote it by u_k.

Theorem 6.6 shows that there exists a homomorphism θ_0 from $\mathcal{C}_0(\Omega)$ into a radical Banach algebra R_0 such that $\ker \theta_0 = \bigcap \mathfrak{P}_0$. Similarly, for each $1 \le k \le n$, there exists a homomorphism θ_k from M_k into R_k such that $\ker \theta_k = \bigcap \mathfrak{P}_k$; where M_k is the maximal modular ideal containing Q_k. We extend θ_k to a homomorphism from $\mathcal{C}_0(\Omega)$ into $R^{\#}$ by setting $\theta_k(u_k) = \mathbf{e}_{R_k}$. It still remains that $\ker \theta_k = \bigcap \mathfrak{P}_k$.

It follows from the result of Bade and Curtis that $\mathcal{I}(\theta_k) = \ker \theta_k$ $(0 \le k \le n)$. Thus the homomorphism $\theta : \mathcal{C}_0(\Omega) \to \prod_{k=0}^n R_k^{\#}$ defined by $\theta(a) = (\theta_0(a), \ldots, \theta_n(a))$ satisfies $\mathcal{I}(\theta) = \bigcap_{k=0}^n \mathcal{I}(\theta_k) = \bigcap \mathfrak{P} = I$. □

Remark. In Parts (ii) of Theorems 6.6 and 6.7, we only need that I is the intersection of a relatively compact family $\mathfrak{P}$ of prime ideals where every chain in the closure of $\mathfrak{P}$ is countable (see Proposition 6.2).

7. Examples on metrizable locally compact spaces

For examples of pseudo-finite sequence of prime ideals in $\mathcal{C}_0(\Omega)$ for Ω metrizable, see [17]. Obviously, unions of finitely many pseudo-finite families are relatively compact. In this section, we shall construct relatively compact families of prime ideals that are not unions of finitely many pseudo-finite families.

Let κ be a well-ordered set. Set $\kappa^{(0)} = \kappa$. For each $n \in \mathbb{N}$, define inductively $\kappa^{(n)}$ as the set of limiting elements in $\kappa^{(n-1)}$. *We shall only consider those κ for which $\kappa^{(n)} = \emptyset$ for some $n \in \mathbb{N}$.* This condition force κ to be countable. Lets call the largest integer d for which $\kappa^{(d)} \ne \emptyset$ the *depth* of κ. *For simplicity, we also suppose that κ has the largest element, $\max \kappa$, and that $\kappa^{(d)} = \{\max \kappa\}$.* Otherwise, we can always replace κ by a bigger well-ordered set.

For each $\alpha \in \kappa$ define $l(\alpha)$ to be the largest integer l for which $\alpha \in \kappa^{(l)}$. We define a partial order $\prec$ on κ as follows: For each $\alpha, \beta \in \kappa$, we write $\alpha \prec \beta$ if β is the smallest element in κ with the properties that $\beta \ge \alpha$ and that $l(\beta) = l(\alpha) + 1$. We define another partial order $\ll$ on κ as follows: For each $\alpha, \beta \in \kappa$, we write $\alpha \ll \beta$ if there exists a finite sequence $\alpha = \gamma_1 \prec \gamma_2 \prec \ldots \prec \gamma_n = \beta$. *Note that* if $\beta \ll \alpha$ and $\beta \le \gamma \le \alpha$ then $\gamma \ll \alpha$.

Lemma 7.1. *Let A be a commutative algebra and Q be an ideal which either is prime in A or is A itself. Suppose that we have $(f_\alpha : \alpha \in \kappa) \subset Q$ and a semiprime ideal $I \subset Q$ such that*

(i) *$f_\alpha \notin I$ and $I{:}f_\alpha \subset Q$;*
(ii) *$f_\alpha f_\beta \in I$ whenever both $\alpha \not\ll \beta$ and $\beta \not\ll \alpha$;*
(iii) *if $gf_\alpha \in I$ then $gf_\beta \in I$ for all $\beta \ll \alpha$;*
(iv) *if $l(\beta_0) = 0$, $\beta_0 \ll \alpha$, $\beta_0 \ne \alpha$ and $gf_{\beta_0} \in I$ then there exists $\beta_1 \ll \alpha$ such that $l(\beta_1) = 0$ and that $gf_\beta \in I$ for all $\beta_1 \le \beta \le \alpha$ with $l(\beta) = 0$.*

Then there exist prime ideals $(P_\alpha : \alpha \in \kappa)$ satisfying that:

(a) *$f_\alpha \notin P_\alpha$ and $I{:}f_\alpha \subset P_\alpha \subset Q$;*
(b) *$P_\alpha = \bigcup_{\beta \ll \alpha, \beta \ne \alpha} P_\beta$;*
(c) *if $g \in P_\alpha$ then there exists $\beta_1 \ll \alpha$ such that $l(\beta_1) = 0$ and that $g \in P_\beta$ for all $\beta_1 \le \beta \le \alpha$ with $l(\beta) = 0$.*

Proof. We prove by induction on the depth d of κ.

When $d=0$, $\kappa=\{0\}$. The conditions (i)-(iv) reduce to I being semiprime, $f_0 \notin I$ and $I{:}f_0 \subset Q$. It follows that $I \cap \left\{f_0^k, f_0^k f :\ k \geq 1, f \in A \setminus Q\right\} = \emptyset$. Therefore, we can find a prime ideal P_0 such that $I_0 \subset P_0 \subset Q$ and $f_0 \notin P_0$. We see that P_0 is the required prime ideal.

Now, suppose that the result hold for all the depth less than d. By Zorn's lemma, we can choose a semiprime ideal J containing I such that J is maximal with respect to conditions (i)-(iv).

Claim 1: If $f \notin Q$ then $J{:}f = J$. Indeed, it is easy to see that $J{:}f$ is semiprime and satisfies conditions (i)-(iv). So the maximality of J implies $J{:}f = J$.

Claim 2: If $f \notin J$ then $J{:}f \subset Q$. For otherwise, there would exist $g \in J{:}f \setminus Q$, and so $f \in J{:}g = J$, by Claim 1; a contradiction.

Set $P = \bigcup_{\alpha\in\kappa} J{:}f_\alpha$. Then $P \subset Q$. Condition (iii) implies that

$$P = \bigcup_{\alpha\in\kappa,\ l(\alpha)=0} J{:}f_\alpha$$

and condition (iv) implies that P is an ideal.

Claim 3: If $f \notin P$ then $J{:}f = J$. Indeed, it is easy to see that $J{:}f$ is semiprime and satisfies conditions (i)-(iv) (the less obvious one is (i), however, since $f \notin P$, $ff_\alpha \notin J$, and so $f_\alpha \notin J{:}f$ and $(J{:}f){:}f_\alpha = J{:}ff_\alpha \subset Q$ by Claim 2). So the maximality of J implies $J{:}f = J$.

Claim 4: P is either prime in A or A itself. Indeed, if $f, g \notin P$, then, by Claim 3,

$$g \notin \bigcup_{\alpha\in\kappa} J{:}f_\alpha = \bigcup_{\alpha\in\kappa} (J{:}f){:}f_\alpha = P{:}f.$$

Thus $fg \notin P$.

Let $\alpha_1 < \alpha_2 < \dots$ be the non-limiting elements in $\kappa^{(d-1)}$; their limit is $\max\kappa$. Set $\kappa_1 = \{\alpha \in \kappa :\ \alpha \leq \alpha_1\}$, and, for each $n \geq 2$, set $\kappa_n = \{\alpha \in \kappa :\ \alpha_{n-1} < \alpha \leq \alpha_n\}$. Each κ_n has depth $d-1$, and $\kappa = \bigcup_{n=1}^\infty \kappa_n \cup \{\max\kappa\}$.

For each $n \in \mathbb{N}$, we see that $(f_\alpha :\ \alpha \in \kappa_n)$, J, and P satisfy $(f_\alpha :\ \alpha \in \kappa_n) \subset P$, $J \subset P$, and conditions (i)-(iv) (with κ_n replacing κ, J replacing I, and P replacing Q). So, by induction, there exist prime ideals P_α ($\alpha \in \kappa_n$) satisfying the conditions (a)-(c) (with obvious modification). Set $P_{\max\kappa} = P$.

Note that if $\beta \ll \alpha < \max\kappa$ then both α and β belong to the same κ_n for some $n \in \mathbb{N}$. We see that the combined sequence $(P_\alpha :\ \alpha \in \kappa)$ obviously satisfies the conditions (a)-(c) (with J replacing I); the only one need to really check is condition (c) when $\alpha = \max\kappa$, however, this case follows from the facts that $J{:}\beta \subset P_\beta \subset P_{\max\kappa}$, that

$$P_{\max\kappa} = \bigcup_{\beta\in\kappa,\ l(\beta)=0} J{:}f_\beta,$$

and that J satisfies condition (iv). □

Now, let Ω be a metrizable locally compact space. We define a non-increasing sequence $\left(\partial^{(n)}\Omega^\flat : n \in \mathbb{Z}^+\right)$ of compact subsets of $\Omega^\flat$ as follows:

(i) put $\partial^{(0)}\Omega^\flat = \Omega^\flat$;

(ii) for each $n \in \mathbb{Z}^+$, define $\partial^{(n+1)}\Omega^\flat$ to be the set of all limit points of $\partial^{(n)}\Omega^\flat$.

Define $\partial^{(\infty)}\Omega^\flat = \bigcap\left\{\partial^{(n)}\Omega^\flat :\ n \in \mathbb{Z}^+\right\}$. By the compactness, either $\partial^{(\infty)}\Omega^\flat$ is non-empty or $\partial^{(l)}\Omega^\flat$ is empty for some $l \in \mathbb{Z}^+$.

In what follows, the hypothesis that $p \in \partial^{(\infty)}\Omega^\flat$ is necessary, because of [17, Proposition 8.7].

Theorem 7.2. *Let Ω be a metrizable locally compact space, and let $p \in \partial^{(\infty)}\Omega^\flat$. Let κ be a well-ordered set as above. Then, there exist a sequence of prime ideals $(P_\alpha :\ \alpha \in \kappa)$ in $\mathcal{C}_0(\Omega)$, where each ideal is supported at p, and a sequence of functions $(f_\alpha :\ \alpha \in \kappa)$ in $\mathcal{C}_0(\Omega)$ satisfying the following:*

(a) *$f_\alpha \notin P_\alpha$ and $f_\beta \in P_\alpha$ whenever both $\beta \not\ll \alpha$ and $\alpha \not\ll \beta$;*
(b) *$P_\alpha = \bigcup_{\beta \ll \alpha, \beta \neq \alpha} P_\beta$;*
(c) *if $g \in P_\alpha$ then there exists $\beta_1 \ll \alpha$ such that $l(\beta_1) = 0$ and that $g \in P_\beta$ for all $\beta_1 \leq \beta \leq \alpha$ with $l(\beta) = 0$.*

Proof. Set $\kappa_0 = \{\beta \in \kappa :\ l(\beta) = 0\}$. For each $\beta \in \kappa \setminus \{\max \kappa\}$, there exists a unique $\alpha \in \kappa$ such that $\beta \prec \alpha$; the set $\{\gamma :\ \gamma \prec \alpha\}$ is order isomorphic to $\mathbb{N}$, and so we can define $t(\beta)$ to be the natural number corresponding to β. For each $\beta \in \kappa_0$, there exists a unique sequence $(\alpha_1, \ldots, \alpha_{d-1}) \in \kappa$ such that

$$\beta \prec \alpha_1 \prec \cdots \prec \alpha_{d-1} \prec \max \kappa;$$

set $w(\beta) = \max \{t(\beta), t(\alpha_1), \ldots, t(\alpha_{d-1})\}$. Note that, for each $k \in \mathbb{N}$,

$$|\{\alpha \in \kappa_0 :\ w(\alpha) \leq k\}| = k^d.$$

Adjoin ∞ to $\mathbb{N}$ to obtain its one-point compactification $\mathbb{N}^\flat$. Define Ξ to be the subset of the product space $(\mathbb{N}^\flat)^{\kappa_0}$ consisting of all elements $(n_\alpha :\ \alpha \in \kappa_0)$ with the properties that there exists a finite set $F \subset \kappa_0$ such that $n_\alpha = \infty$ $(\alpha \in \kappa_0 \setminus F)$ and such that $n_\alpha \geq \max \{w(\beta) : \beta \in F\}$ $(\alpha \in \kappa_0)$. It is easy to see that Ξ is a closed subset of $(\mathbb{N}^\flat)^{\kappa_0}$. Since κ must be countable, the space Ξ can be embedded into $\Omega^\flat$ such that $\boldsymbol{\infty} = (\infty, \infty, \ldots)$ is mapped into p (similar to the proofs of [17, Lemmas 9.1 and 9.2]). Thus we only need to find a system of prime ideals $(P_\alpha :\ \alpha \in \kappa)$ and functions $(f_\alpha :\ \alpha \in \kappa)$ satisfying condition (a)-(c) in $\mathcal{C}(\Xi)$ such that all P_α are supported at $\boldsymbol{\infty}$.

For each $\alpha \in \kappa_0$, define

$$Z_\alpha = \{(j_\beta)_{\beta \in \kappa_0} \in \Xi :\ j_\alpha = \infty\},$$

and for $\alpha \in \kappa \setminus \kappa_0$, define

$$Z_\alpha = \bigcap_{\beta \ll \alpha,\ l(\beta)=0} Z_\beta.$$

Then choose $f_\alpha \in \mathcal{C}(\Xi)$ such that $Z_\alpha = \mathbf{Z}(f_\alpha)$. Let $\mathcal{F}$ to be the z-filter generated by all $Z_\alpha \cup Z_\beta$ $(\alpha, \beta \in \kappa,\ \alpha \not\ll \beta$ and $\beta \not\ll \alpha)$. Then define $I = \mathbf{Z}^{-1}[\mathcal{F}]$. Obviously I is a semiprime ideal, $I \subset M_{\boldsymbol{\infty}}$, and $(f_\alpha :\ \alpha \in \kappa) \subset M_{\boldsymbol{\infty}}$.

We shall prove that (f_α), I, and $M_{\boldsymbol{\infty}}$ satisfy conditions (i)-(iv) of Lemma 7.1. The result then follows by applying that lemma.

First, for each $\gamma \in \kappa$, $f \in I{:}f_\gamma$ if and only if

$$\mathbf{Z}(f) \cup Z_\gamma \supset \bigcap_{k=1}^{n} (Z_{\alpha_k} \cup Z_{\beta_k}),$$

where, for each k, $\alpha_k \not\ll \beta_k$ and $\beta_k \not\ll \alpha_k$. We see that, for each k, one of the following three cases must happen: (1) $\alpha_k \not\ll \gamma$ and $\gamma \not\ll \alpha_k$, (2) $\gamma \not\ll \beta_k$ and

$\beta_k \not\ll \gamma$, (3) $\alpha_k \ll \gamma$, $\beta_k \ll \gamma$, $\alpha_k \not\ll \beta_k$ and $\beta_k \not\ll \alpha_k$. Thus, we see that

$$\mathbf{Z}(f) \cup Z_\gamma \supset \bigcap_{i=1}^{r} Z_{\varrho_i} \cap \bigcap_{j=1}^{s} \left(Z_{\sigma_j} \cup Z_{\varsigma_j} \right),$$

where, for each i, $\varrho_i \not\ll \gamma$ and $\gamma \not\ll \varrho_i$, and for each j, $\sigma_j \ll \gamma$, $\varsigma_j \ll \gamma$, $\sigma_j \not\ll \varsigma_j$ and $\varsigma_j \not\ll \sigma_j$. It follows that Lemma 7.1(i) holds: $f_\gamma \notin I$ and $I{:}f_\gamma \subset M_{\infty}$ $(\gamma \in \kappa)$. It is easily seen that conditions (ii) and (iii) of Lemma 7.1 are satisfied by the definitions of I and Z_α's.

Now, let $\beta_0, \alpha \in \kappa$ and let $g \in \mathcal{C}(\Xi)$ be such that $l(\beta_0) = 0$, $\beta_0 \ll \alpha$, $\beta_0 \neq \alpha$ and $gf_{\beta_0} \in I$ then, from the previous discussion, noting that $l(\beta_0) = 0$, we have

$$\mathbf{Z}(g) \cup Z_{\beta_0} \supset \bigcap_{i=1}^{r} Z_{\varrho_i};$$

where, for each i, $\beta_0 \not\ll \varrho_i$ (and so $\alpha \not\ll \varrho_i$). This implies that

$$\mathbf{Z}(g) \supset \bigcap_{i=1}^{r} Z_{\varrho_i} \cap \bigcap_{l(\gamma)=0,\, w(\gamma) \le w(\beta_0)} Z_\gamma .$$

Without loss of generality, we assume that $\varrho_i \ll \alpha$ $(1 \le i \le k)$ and $\varrho_i \not\ll \alpha$ $(k < i \le r)$. Choose β_1 to be the smallest element in κ with respect to the properties that $l(\beta_1) = 0$, $\beta_1 < \alpha$, $\beta_1 > \max\{\varrho_i : 1 \le i \le k\}$, and that

$$\beta_1 > \max\{\gamma \in \kappa_0 : \ \gamma \ll \alpha \text{ and } w(\gamma) \le w(\beta_0)\}$$

(it is easy to see that there exists such an element). It follows that $\beta_1 \ll \alpha$. Let $\beta \in \kappa_0$ be such that $\beta_1 \le \beta \le \alpha$. We see that $\beta \ll \alpha$, and so $\varrho_i \not\ll \beta$ (otherwise, $\varrho_i = \beta \ll \alpha$), and also $\beta \not\ll \varrho_i$ (otherwise, either $\varrho_i \ll \alpha$ or $\alpha \ll \varrho_i$) for $1 \le i \le r$. Obviously, $\beta \not\ll \gamma$ and $\gamma \not\ll \beta$ for each $\gamma \in \kappa_0$ with $w(\gamma) \le w(\beta_0)$. Thus

$$\mathbf{Z}(g) \cup Z_\beta \supset \bigcap_{i=1}^{r} (Z_{\varrho_i} \cup Z_\beta) \cap \bigcap_{l(\gamma)=0,\, w(\gamma) \le w(\beta_0)} (Z_\gamma \cup Z_\beta) \in \mathcal{F};$$

and so $gf_\beta \in I$. Hence, Lemma 7.1 (iv) holds. This finishes the proof. $\square$

Remark. Since the cardinality of $\mathcal{C}(\Xi)$ is $\mathfrak{c}$, we see that

$$\left| \mathcal{C}_0(\Omega) \Big/ \bigcap_{\alpha \in \kappa} P_\alpha \right| = \mathfrak{c}.$$

Proposition 7.3. *The family of prime ideals $\{P_\alpha : \ \alpha \in \kappa\}$ constructed in Theorem 7.2 is compact. If moreover the depth d of κ is bigger than 1, then $\bigcap_{\alpha \in \kappa} P_\alpha$ is not the intersection of any family that can be decomposed into only finitely many pseudo-finite subfamilies of prime ideals.*

Proof. Consider the order topology on κ. Then κ is a compact metrizable space. Each sequence in κ has a convergence subsequence. The following claim will show the compactness of $\{P_\alpha : \ \alpha \in \kappa\}$.

Claim: For each sequence (α_n) converging to α, there exists n_0 such that $(P_{\alpha_n} : n \ge n_0)$ is a pseudo-finite sequence whose union is P_α. Proof: Without loss of

generality, we assume that $\alpha_n \neq \alpha$ ($n \in \mathbb{N}$). There exists n_0 such that $\alpha_n \ll \alpha$ ($n \geq n_0$). We see that $P_{\alpha_n} \subset P_\alpha$ ($n \geq n_0$) and

$$P_\alpha = \bigcup_{\beta \ll \alpha,\ l(\beta)=0} P_\beta.$$

So, for each $g \in P_\alpha$, by Theorem 7.2(c), there exists $\beta_1 \ll \alpha$ such that $l(\beta_1) = 0$ and that $g \in P_\beta$ for all $\beta_1 \leq \beta \leq \alpha$ with $l(\beta) = 0$. Choose n_1 such that $\alpha_n \geq \beta_1$ ($n \geq n_1$). For each $n \geq n_1$, pick $\beta' \ll \alpha_n$ and $l(\beta') = 0$. If $\beta' < \beta_1$, then $\beta_1 \ll \alpha_n$, and so $g \in P_{\beta_1} \subset P_{\alpha_n}$. Otherwise, $\beta_1 \leq \beta'$, then $g \in P_{\beta'} \subset P_{\alpha_n}$. Thus, $(P_{\alpha_n} :\ n \geq n_0)$ is a pseudo-finite sequence whose union is P_α.

Suppose that $I = \bigcap_{\alpha \in \kappa} P_\alpha$ is the intersection of a family $\mathfrak{P}$ that can be decomposed into finitely many pseudo-finite subfamilies of prime ideals. By keeping only the minimal elements of $\mathfrak{P}$, we can assume that $P \not\subset Q$ for each $P \neq Q \in \mathfrak{P}$. Lemma 5.6 shows that, for each $P \in \mathfrak{P}$ there exists $f_P \notin P$ but $f_P \in Q$ for all $Q \in \mathfrak{P} \setminus \{P\}$. This and Lemma 4.8 then implies that $\mathfrak{P}$ is the set of prime ideals of the form $I{:}f$ for some $f \in \mathcal{C}_0(\Omega)$. Similarly, $\{P_\alpha :\ \alpha \in \kappa,\ l(\alpha) = 0\}$ is the set of prime ideals of the form $I{:}f$ for some $f \in \mathcal{C}_0(\Omega)$. Thus

$$\mathfrak{P} = \{P_\alpha :\ \alpha \in \kappa,\ l(\alpha) = 0\},$$

and obviously this cannot be the union of any finite number of pseudo-finite families when the depth of κ is bigger than 1. $\square$

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